PG – 208

II Semester M.Sc. Degree Examination, July 2017
(CBCS)
MATHEMATICS
M205T : Functional Analysis

Time : 3 Hours   Max. Marks : 70

Instructions : *1) Answer **any five full** questions.*
*2) **All** questions carry **equal** marks.*

1. a) State and prove
   i) Cauchy-Schwartz inequality.
   ii) Minkowski inequality.
   b) Show that the collection B(x) of all bounded functions on a set X is a complete normed linear space.
   c) Show that if $T : N \to N'$ is a linear transformation between normed linear spaces N and N' then prove that the following are equivalent.
   i) T is continuous at the origin.
   ii) T is bounded. **(5+4+5)**

2. a) If M is a closed subspace of a normed linear space N then show that there is a natural map $T : N \to \frac{N}{M}$ which is linear and continuous with $\| T \| \leq 1$.
   b) State and prove Hahn Banach theorem for both real and complex cases. **(4+10)**

3. a) If N is a normed linear space and $x_0 \in N$ with $x_0 \neq 0$ then show that there exists a functional $f_0 \in N^*$ such that $f_0(x_0) = \| x_0 \|$ and $\| f_0 \| = 1$.
   b) If M is a closed linear subspace of a normed linear space N and $x_0 \notin M$ then show that there is a functional $h \in N^*$ such that $h(x_0) = 1$ and $h(M) = 0$ with $\| h \| = \frac{1}{d}$, where $d = d(x_0, M)$. **(7+7)**

P.T.O.

4. a) State the open mapping theorem. Use it to prove that a one-one linear continuous map T from a Banach space B onto a Banach space B' is a homeomorphism.

   b) State and prove closed graph theorem.

   c) Let N be a normed linear space and X be a non empty subset of N. If f(x) is bounded for eacg $f \in N^*$ then show that X is bounded. **(3+8+3)**

5. a) Prove the Parallelogram law and polarisation identity in a Hilbert space.

   b) Show that orthogonal complement of a subset of a Hilbert space H is a closed linear subspace of H.

   c) Prove that translation on a Hilbert space is a homeomorphism. **(5+4+5)**

6. a) Let M be a closed linear subspace of a Hilbert space H and let $x \notin M$ and d = d(X, M). Then prove that there is a unique vector $y_0 \in M$ such that $\| x - y_0 \| = d$

   b) If M is a closed linear subspace of a Hilbert space H then prove that $H = M \oplus M^{\perp}$.

   c) If $\{e_1, e_2, ..., e_n\}$ is a finite orthonormal set in a Hilbert space H and $x \in H$ then prove that

   i) $\sum_{i=1}^{n} |\langle x, e_i \rangle|^2 \le \| x \|^2$. ii) $x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \perp e_j, \forall j$. **(5+4+5)**

7. a) If T is an operator on a Hilbert space H then prove that

   T = 0 if and only if $\langle Tx, y \rangle = 0 \; \forall \; x, y \in H$.

   b) Show that the collection of all self adjoint operators on a Hilbert space H form a closed real linear subspace of H.

   c) Show that eigen values of a self adjoint operator are real and the eigen vectors associated with distinct eigen values are orthogonal. **(3+4+7)**

8. a) If N_1 and N_2 are two normal operators on a Hilbert space H such that either commutes with the adjoint of the other then prove that the product and the sum of N_1 and N_2 are also normal operators.

   b) Define an Unitary operator. Show that an Unitary operator preserves the inner product and the norm.

   c) Show that if a normal operator T has eigen value λ then its adjoint T^* has eigen value $\bar{\lambda}$. **(5+6+3)**
